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Cambridge International AS & A Level

Further Mathematics

Coursebook

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Further Pure Mathematics 2

Introduction

Cambridge International AS & A Level Further Mathematics is a very rigorous and rewarding course that builds on top of the A Level Mathematics course. The Further Mathematics course is designed for students who wish to understand mathematics at a much higher level, and who have already successfully completed the A Level Mathematics course, although with careful planning it can also be studied alongside A Level Mathematics.

The course is divided up into three major areas: Pure Mathematics, Statistics and Mechanics. There are 13 Pure Mathematics topics, 5 Statistics topics and 6 Mechanics topics, which make up the 4 examination papers that are available to students. Due to the flexible nature of the modules, students can take either AS Further Mathematics or A Level Further Mathematics. The 24 topics build on knowledge already obtained in the A Level Mathematics course.

This coursebook has been written to reflect the rigour and flexibility of the Further Mathematics course. The authors have almost 30 years of Further Mathematics teaching between them, and have used their experience to create a comprehensive and supportive companion to the course. While the majority of the examples are within the scope of the course, there are opportunities, discussions and examples that will stretch the curious mind.

The book is designed not only to instruct students what is required, but also to help students develop their own understanding of important concepts. Frequent, detailed worked examples guide students through the steps in a solution, and numerous practice questions and past paper questions provide opportunities for students to apply their learning. In addition, there are larger review exercises and four practice exam-style papers for students to practise and consolidate what they have covered during the course. The questions have been written to provide a rich and diverse approach to solving problems with the intention of enhancing deep learning. Every care has been taken to ensure that the English used in this book is accessible to students with English as an additional language. This is supported by a glossary of the key terms essential to the course.

This book is the first of its kind, and the authors are confident that it will support both students and teachers to master the course.

The authors wish you the very best in your undertaking of this course.

Lee Mckelvey
Martin Crozier

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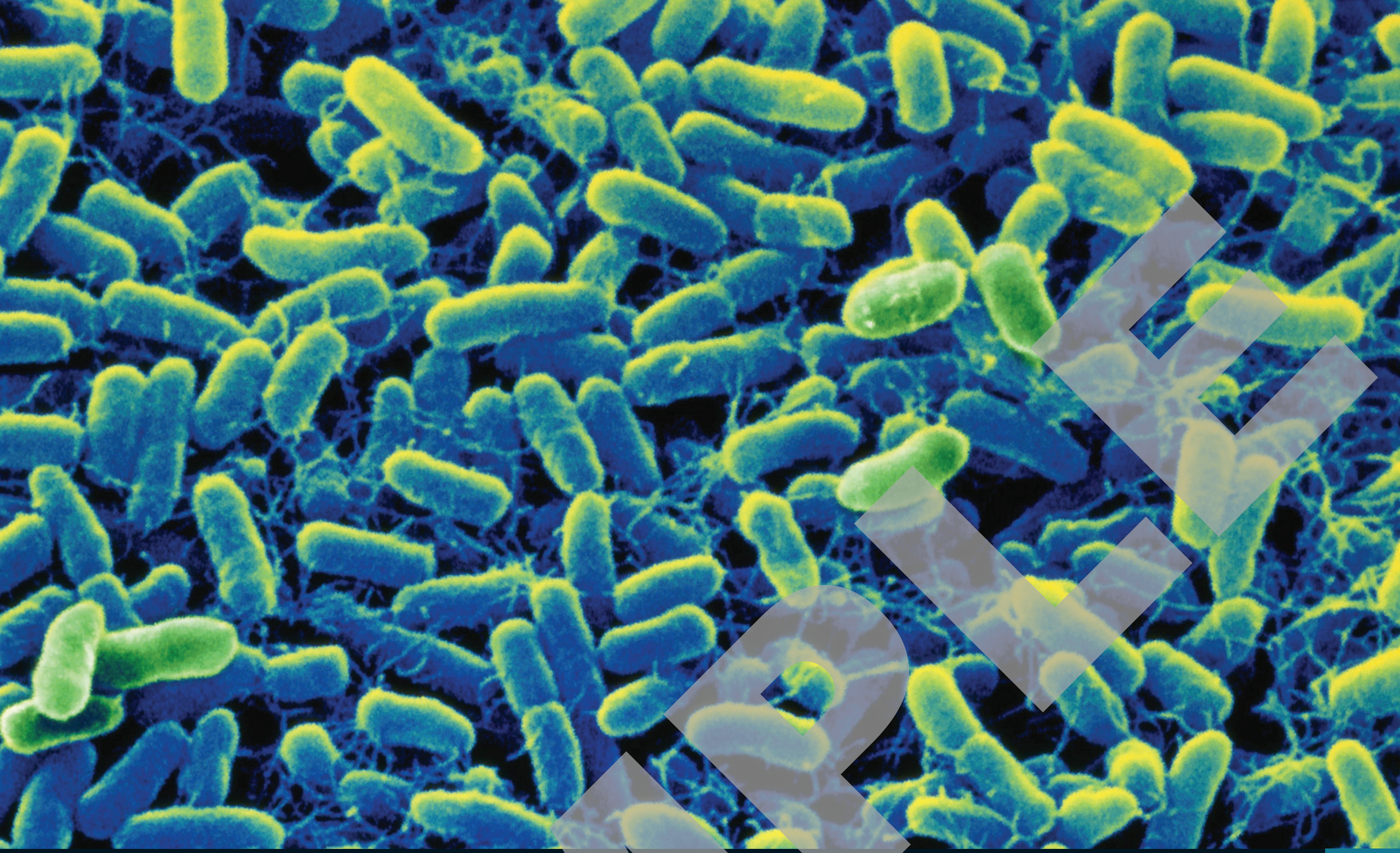

Chapter 1
Roots of polynomial equations

In this chapter you will learn how to:

- recall and use the relations between the roots and coefficients of polynomial equations
- use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation.

PREREQUISITE KNOWLEDGE

Where it comes from	What you should be able to do	Check your skills
AS & A Level Mathematics Pure Mathematics 1	Use simple substitutions to make another variable the subject.	**1** Rewrite the following equations in terms of the new variable. **a** $x^2 - 3x + 5 = 0, y = x - 2$ **b** $x^3 + 2x^2 - 4 = 0, y = \frac{2}{x}$ **c** $x^3 - 3x + 7 = 0, y = \frac{1}{x+1}$
AS & A Level Mathematics Probability & Statistics 1	Work with basic sigma notation, such as Σx and Σx^2.	**2** Evaluate the following. **a** $\sum_{r=1}^{10} r$ **b** $\sum_{r=1}^{10} 3$ **c** $\sum_{r=1}^{10} (r+2)$
AS & A Level Mathematics Pure Mathematics 1	Work with basic recurrence relations.	**3** Write the first 6 terms for the following relations. **a** $u_{n+1} = 3u_n + 2, u_1 = 1$ **b** $u_{n+2} = 2u_{n+1} - u_n + 5, u_1 = 1, u_2 = 1$

What are polynomials?

Polynomials are algebraic expressions made up of several variables and a sum of multiples of non-negative integer powers of variables. For example, $2x^2 - 3xy + 5x$ is a polynomial, but neither $3x^{\frac{1}{2}}$ nor $\frac{5}{y}$ are polynomials. Engineers use them to ensure that a new building can withstand the force of an earthquake. Medical researchers use them to model the behaviour of bacterial colonies.

We already know how to divide a polynomial by a linear term and identify the quotient and any remainder. We have worked with simpler polynomials when completing the square of a quadratic or finding the discriminant. Now we will extend this knowledge to work with higher powers. We will also use algebraic manipulation to understand the conditions for complex solutions and to combine polynomials with summation notation and recurrence relations.

In this chapter, we will look at ways to find characteristics of polynomials, finding the sum and product of roots, as well as other properties linked to their roots.

1.1 Quadratics

To begin with, let us look back at the **quadratic** equation $ax^2 + bx + c = 0$. If we write this in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then we can compare it to the form $(x - \alpha)(x - \beta) = 0$. This shows that the sum of the **roots** is $\alpha + \beta = -\frac{b}{a}$, and the product of the roots is $\alpha\beta = \frac{c}{a}$, as shown in Key point 1.1. Hence, we can say that $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

KEY POINT 1.1

If we write a quadratic in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, the sum of the roots is $\alpha + \beta = -\frac{b}{a}$.

The product of the roots of the quadratic is $\alpha\beta = \frac{c}{a}$.

WORKED EXAMPLE 1.1

The quadratic equation $x^2 - 2px + p = 0$ is such that one root is three times the value of the other root. Find p.

Answer

$\alpha + 3\alpha = 2p$ — Using $\alpha + \beta = -\frac{b}{a}$.

$p = 2\alpha$

$\alpha \times 3\alpha = p$ — Using $\alpha\beta = \frac{c}{a}$.

$p = 3\alpha^2$ — Equate the two results.

$\frac{p}{3} = \left(\frac{p}{2}\right)^2$ — Cross multiply.

$4p - 3p^2 = 0$ — Factorise and omit the case when $p = 0$.

$p = \frac{4}{3}$

Using $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$, we can begin to define many other results, but first we must introduce some new notation. The sum of the roots can be written as $\Sigma\alpha = \alpha + \beta$ and the product can be written as $\Sigma\alpha\beta = \alpha\beta$.

Let us consider how to determine the value of $\alpha^2 + \beta^2$. The natural first step is to expand $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$. Hence, we can say that $\alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$. We denote $\alpha^2 + \beta^2$ as $\Sigma\alpha^2$.

Next, look at $(\alpha - \beta)^2$. Again, expanding the brackets is a good start. So $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$. Hence, we can see that $(\alpha - \beta)^2 = \Sigma\alpha^2 - 2\Sigma\alpha\beta$.

We can write $\frac{1}{\alpha} + \frac{1}{\beta}$ as $\Sigma\frac{1}{\alpha}$. How do we find the sum of $\frac{1}{\alpha} + \frac{1}{\beta}$? First, combine the two fractions to get $\frac{\alpha + \beta}{\alpha\beta}$. We can see that this is $\frac{\Sigma\alpha}{\Sigma\alpha\beta}$. Similarly, we can write $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ as $\Sigma\frac{1}{\alpha^2}$ and we can show $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\Sigma\alpha^2}{(\Sigma\alpha\beta)^2}$.

TIP

Note the difference between $(\Sigma\alpha)^2$ and $\Sigma\alpha^2$.

WORKED EXAMPLE 1.2

Find $\alpha^3 + \beta^3$ in summation notation.

Answer

$(\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ — Use the binomial theorem $(x + y)^n$.

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ — Rearrange and factorise.

$\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha$ — Sum for each possible root.

Alternatively, use $-3\Sigma\alpha^2\beta$ in place of $-3\Sigma\alpha\beta\Sigma\alpha$. However, it is not as easy to calculate with this form.

Some of the results found can be written in alternative forms, using a recurrence relation such as $S_n = \alpha^n + \beta^n$. If we consider the quadratic $x^2 + 5x + 7 = 0$, we can see that $\alpha + \beta = -5$. This result can also be viewed as $S_1 = \alpha + \beta = -5$. To determine the value of $\alpha^2 + \beta^2$, we can approach this from another angle.

Given that α and β are roots of the original equation, we can state that $\alpha^2 + 5\alpha + 7 = 0$ and $\beta^2 + 5\beta + 7 = 0$. Adding these together gives the result $(\alpha^2 + \beta^2) + 5(\alpha + \beta) + 14 = 0$ or $S_2 + 5S_1 + 14 = 0$. Now we can work out the value of S_2 or $\alpha^2 + \beta^2$. From $S_2 + 5S_1 + 14 = 0$ and $S_1 = -5$ we have $S_2 = \alpha^2 + \beta^2 = 11$. Note this could also have been found from $\alpha^2 + \beta^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta = (-5)^2 - 2(7) = 11$.

WORKED EXAMPLE 1.3

Given that $2x^2 + 3x - 2 = 0$ has roots α, β, find the values of $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$.

Answer

$2\alpha^2 + 3\alpha - 2 = 0$

$2\beta^2 + 3\beta - 2 = 0$

$\Rightarrow 2S_2 + 3S_1 - 4 = 0$ — Add both equations to get the recurrence form.

$S_1 = -\frac{3}{2}$ — State $S_1 = -\frac{b}{a}$ from the original quadratic.

$S_2 = \alpha^2 + \beta^2 = \frac{17}{4}$ — Substitute the S_1 value into the equation.

$2x^2 + 3x - 2 = 0$

$\Rightarrow 2x^3 + 3x^2 - 2x = 0$ — Multiply by x.

$\Rightarrow 2S_3 + 3S_2 - 2S_1 = 0$ — Add $2\alpha^3 + 3\alpha^2 - 2\alpha = 0$ and $2\beta^3 + 3\beta^2 - 2\beta = 0$.

$S_3 = \alpha^3 + \beta^3 = -\frac{63}{8}$ — Use the values of S_1 and S_2.

EXERCISE 1A

M 1 Each of the following quadratic equations has roots α, β. Find the values of $\alpha + \beta$ and $\alpha\beta$.

a $x^2 + 5x + 9 = 0$ b $x^2 - 4x + 8 = 0$ c $2x^2 + 3x - 7 = 0$

M 2 Given that $3x^2 + 4x + 12 = 0$ has roots α, β, find:

a $\alpha + \beta$ and $\alpha\beta$ b $\alpha^2 + \beta^2$

M 3 $x^2 - (2 + p)x + (7 + p) = 0$ has roots that differ by 1. Find the value of p given that $p > 0$.

M 4 If $a + b = -3$ and $a^2 + b^2 = 7$, find the value of ab and, hence, write down a quadratic equation with roots a and b.

P 5 If $x^2 + bx + c = 0$ has roots α and β, prove that:

a If $\alpha = 3\beta$, then $b^2 = \frac{16}{3}c$.

b If $\alpha = \beta - 2$, then $b^2 = 4(c + 1)$.

M 6 You are given the quadratic equation $px^2 + qx - 16 = 0$ which has roots α and β. Given also that $\alpha + \beta = -\frac{1}{2}$ and $\alpha\beta = -8$, find the values of p and q.

M 7 The quadratic equation $x^2 + 2x - 6 = 0$ has roots α and β. Find the values of $(\alpha - \beta)^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

M 8 A quadratic equation has roots α and β. Given that $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{2}$ and $\alpha^2 + \beta^2 = 12$, find two possible quadratic equations that satisfy these values.

M 9 The quadratic equation $3x^2 + 2x - 4 = 0$ has roots α and β. Find the values of S_1, S_2 and S_{-1}.

10 You are given the quadratic equation $4x^2 - x + 6 = 0$ which has roots α and β.

M a Find $\alpha^2 + \beta^2$.

PS b Without solving the quadratic equation, state what your value for part **a** tells you about the roots.

1.2 Cubics

In this section we will be looking at **cubic equations**. We will use the same concepts as Section 1.1, but this time the roots will be α, β and γ.

Beginning with $ax^3 + bx^2 + cx + d = 0$, the first step is to divide by the constant a to get $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$.

Next, relate this to $(x - \alpha)(x - \beta)(x - \gamma) = 0$ to establish the relation:

$$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$$

Then $\alpha + \beta + \gamma = -\frac{b}{a}$, which is known as $\Sigma\alpha$ or S_1.

Other results are $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, written as $\Sigma\alpha\beta$, and $\alpha\beta\gamma = -\frac{d}{a}$, written as $\Sigma\alpha\beta\gamma$.

Recall from quadratics that $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$. This is the same result for a cubic, where the term $(\Sigma\alpha)^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$, as shown in Key point 1.2.

TIP

Following on from the idea you saw in Worked example 1.3, if we consider the notation $S_n = \alpha^n + \beta^n + \gamma^n$ and then use it to represent our roots, just as with quadratics, we can use S_2 to represent $\alpha^2 + \beta^2 + \gamma^2$ and so on.

KEY POINT 1.2

$$(\Sigma\alpha)^2 = (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$$

WORKED EXAMPLE 1.4

Find the summation form for the results $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

Answer

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$ — Combine the fractions.

$\Rightarrow \Sigma\frac{1}{\alpha} = \frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$ — State the result.

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2}{\alpha^2\beta^2\gamma^2}$ — Combine the fractions, as before.

$\Sigma\frac{1}{\alpha^2} = \frac{\Sigma(\alpha\beta)^2}{(\Sigma\alpha\beta\gamma)^2}$ — State the result.

All of the results derived for quadratics can also be written for cubics, but the algebra is rather cumbersome. Using three roots can take time to work through. You are encouraged to convince yourself that for a cubic it is true that $\Sigma\alpha^3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma$.

WORKED EXAMPLE 1.5

Given that $x^3 + 2x^2 + 5 = 0$, find, using summation form, the values of S_1, S_2, S_3 and S_{-1}.

Answer

$S_1 = -2$ — Recall this is $-\frac{b}{a} = -\frac{2}{1}$.

$S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ — Recall the value of $\Sigma\alpha\beta$ is given by $\frac{c}{a} = \frac{0}{1}$ as the linear term coefficient is 0.

$\Rightarrow S_2 = (-2)^2 - 2 \times 0 = 4$

$S_3 = (\Sigma\alpha)^3 - 3\Sigma\alpha\beta\Sigma\alpha + 3\Sigma\alpha\beta\gamma$ — The last term is $3\alpha\beta\gamma$.

$S_3 = (-2)^3 - 3 \times (0) \times (-2) + 3 \times (-5)$ — Substitute the values into the equation.

$S_3 = -23$

$S_{-1} = \frac{0}{-5} = 0$ — Recall that this result is equivalent to $\frac{c}{-d}$, which is obtained by taking the negative of the coefficient of the linear term and dividing by the constant term.

Worked example 1.5 uses the summation form, which is rather cumbersome, especially for S_3. Finding higher powers, such as S_4 or S_5, using this method would be very time consuming.

In Worked example 1.6 we will use the recurrence form to evaluate results such as S_3 and S_4.

Before we look at the next example, since α, β, γ all satisfy our cubics, it is true that, for example, for $x^3 + 3x^2 + 6 = 0$ we can see that $\alpha^3 + 3\alpha^2 + 6 = 0, \beta^3 + 3\beta^2 + 6 = 0$ and $\gamma^3 + 3\gamma^2 + 6 = 0$.

Adding them gives $\alpha^3 + \beta^3 + \gamma^3 + 3(\alpha^2 + \beta^2 + \gamma^2) + 18 = 0$ or $S_3 + 3S_2 + 18 = 0$.

WORKED EXAMPLE 1.6

For the cubic equation $3x^3 + 2x^2 - 4x + 1 = 0$, find the value of S_3.

Answer

$S_1 = -\frac{2}{3}$ — From $-\frac{b}{a}$.

$S_2 = \left(-\frac{2}{3}\right)^2 - 2 \times \left(-\frac{4}{3}\right) = \frac{28}{9}$ — Using $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

$3S_3 + 2S_2 - 4S_1 + 3 = 0$ — Note that since $3\alpha^3 + 2\alpha^2 - 4\alpha + 1 = 0$ and similar equations can be made for β, γ, we have +1 three times. β and γ, when we add the three equations, we have +1 three times, so the constant term in the equation on the left becomes +3.

$S_3 = -\frac{107}{27}$ — Substitute for the final result.

We have already seen how to manipulate a polynomial to get a higher power result, such as obtaining S_3 from a quadratic. Imagine we want to obtain a value such as S_{-2} from a cubic equation, using only recurrence methods.

The first step would be to multiply our cubic by x^{-2} to give $ax + b + \frac{c}{x} + \frac{d}{x^2} = 0$. The recurrence formula would then be $aS_1 + 3b + cS_{-1} + dS_{-2} = 0$. Note the constant term, b, is multiplied by 3. Now we need to find only S_1 and S_{-1}, and from the original equation this is straightforward.

WORKED EXAMPLE 1.7

For the cubic equation $x^3 - 3x^2 + 4 = 0$, find the value of S_{-3}.

Answer

$S_{-1} = \frac{0}{-4} = 0$ — Recall that $S_{-1} = \frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$.

$x^3 - 3x^2 + 4 = 0$

$\Rightarrow 1 - \frac{3}{x} + \frac{4}{x^3} = 0$ — Divide by x^3.

$3 - 3S_{-1} + 4S_{-3} = 0$ — Remember that the 1 is counted three times as there are three roots from adding the three equations for α, β and γ.

$S_{-3} = -\frac{3}{4}$ — Substitute values into the equation.

With Worked example 1.7 in mind, note that for a general cubic of the form $ax^3 + bx^2 + cx + d = 0$, if we multiply by x^n then our recurrence formula is $aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$. Note also that, in the absence of any constant terms that are independent of x, they would have their coefficients counted only once.

EXERCISE 1B

M **1** Each of the following cubic equations has roots α, β, γ. Find, for each case, $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

- **a** $x^3 + 3x^2 - 5 = 0$
- **b** $2x^3 + 5x^2 - 6 = 0$
- **c** $x^3 + 7x - 9 = 0$

M **2** Given that $x^3 - 3x^2 + 12 = 0$ has roots α, β, γ, find the following values:

- **a** $\alpha + \beta + \gamma$ and $\alpha\beta + \alpha\gamma + \beta\gamma$
- **b** $\alpha^2 + \beta^2 + \gamma^2$

M **3** The roots of each of the following cubic equations are α, β, γ. In each case, find the values of S_2 and S_{-1}.

- **a** $x^3 - 2x^2 + 5 = 0$
- **b** $3x^3 + 4x - 1 = 0$
- **c** $x^3 + 3x^2 + 5x - 7 = 0$

M **4** The cubic equation $x^3 - x + 7 = 0$ has roots α, β, γ. Find the values of $\Sigma\alpha$ and $\Sigma\alpha^2$.

M **5** Given that $2x^3 + 5x^2 + 1 = 0$ has roots α, β, γ, and that $S_n = \alpha^n + \beta^n + \gamma^n$, find the values of S_2 and S_3.

M PS **6** The cubic equation $x^3 + ax^2 + bx + a = 0$ has roots α, β, γ, and the constants a, b are real and positive.

- **a** Find, in terms of a and b, the values of $\Sigma\alpha$ and $\Sigma\frac{1}{\alpha}$.
- **b** Given that $\Sigma\alpha = \Sigma\frac{1}{\alpha}$, does this cubic equation have complex roots? Give a reason for your answer.

M **7** The cubic equation $x^3 - x + 3 = 0$ has roots α, β, γ.

- **a** Using the relation $S_n = \alpha^n + \beta^n + \gamma^n$, or otherwise, find the value of S_4.
- **b** By considering S_1 and S_4, determine the value of $\alpha^3(\beta + \gamma) + \beta^3(\alpha + \gamma) + \gamma^3(\alpha + \beta)$.

M P **8** A cubic polynomial is given as $2x^3 - x^2 + x - 5 = 0$, having roots α, β, γ.

- **a** Show that $2S_{n+3} - S_{n+2} + S_{n+1} - 5S_n = 0$.
- **b** Find the value of S_{-2}.

M **9** The cubic equation $px^3 + qx^2 + r = 0$ has roots α, β, γ. Find, in terms of p, q, r:

- **a** S_1
- **b** S_2
- **c** S_3

1.3 Quartics

Now that we are working with **quartics**, it is best to use the recurrence formula as often as we can. This is especially true for the sum of the cubes ($= \alpha^3 + \beta^3 + \gamma^3 + \delta^3$). If we want to determine the sum of the cubes of a general quartic, the best way is to first note down S_1, then determine S_2 and S_{-1}. After this, we can then use the form $aS_4 + bS_3 + cS_2 + dS_1 + 4e = 0$, then divide by x to obtain S_3. This process allows us to work out other values, especially those beyond the highest power.

As we have seen with previous polynomials, there are standard results that are defined by observation from previous cases, but the algebra for some results is too complicated to be discussed here.

So, with our roots now being $\alpha, \beta, \gamma, \delta$, we have $\Sigma\alpha = -\frac{b}{a}$, $\Sigma\alpha\beta = \frac{c}{a}$, $\Sigma\alpha\beta\gamma = -\frac{d}{a}$ and $\Sigma\alpha\beta\gamma\delta = \frac{e}{a}$.

In addition to these, we also have $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ and $S_{-1} = \dfrac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$ and so on.

Note again that, algebraically it is much more sensible to use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

When converting a polynomial to our recurrence formula, the constant is always multiplied by a factor of n from the original equation. As an example, $x^4 - 3x^3 - 5 = 0$ would give $S_4 - 3S_3 - 20 = 0$.

WORKED EXAMPLE 1.8

A quartic polynomial is given as $x^4 + 3x^2 - x + 5 = 0$, it has roots $\alpha, \beta, \gamma, \delta$. Find the values of S_2 and S_4.

Answer

$S_1 = 0$	Simply state the negative of the coefficient of x^3, as the coefficient of x^4 is 1.
$S_2 = 0^2 - 2 \times 3 = -6$	Use $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.
$S_4 + 3S_2 - S_1 + 20 = 0$	Use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.
$S_4 = -2$	Final answer.

TIP

Remember that for any polynomial, $\Sigma\frac{1}{\alpha}$ is always obtained using the negative of the coefficient of the linear term over the constant term.

WORKED EXAMPLE 1.9

For the quartic $x^4 - x^3 + 2x^2 - 2x - 5 = 0$, state the values of S_1 and S_{-1}, and determine the value of S_2. State whether or not there are any complex solutions.

Answer

$S_1 = 1$	$-1 \times (-1)$
$S_{-1} = \frac{-(-2)}{-5} = -\frac{2}{5}$	Use $S_{-1} = \frac{\Sigma\alpha\beta\gamma}{\Sigma\alpha\beta\gamma\delta}$.
$S_2 = (1)^2 - 2 \times 2 = -3$	Use $S_2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.
Yes, there are complex solutions.	Since $S_2 < 0$.

TIP

Don't even try to use an algebraic approach to quartics, especially for S_3 and higher. Use the recurrence method.

KEY POINT 1.3

For quartics, use $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$ as a recurrence model to determine higher-powered roots.

EXERCISE 1C

M 1 For each of the following quartic equations, find the values of $\Sigma\alpha$ and $\Sigma\alpha\beta$.

a $x^4 - 2x^3 + 5x^2 + 7 = 0$

b $2x^4 + 5x^3 - 3x + 4 = 0$

c $3x^4 - 2x^2 + 9x - 11 = 0$

M 2 The quartic equation $5x^4 - 3x^3 + x - 13 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find:

a $\Sigma\alpha$ and $\Sigma\alpha^2$

b $\Sigma\frac{1}{\alpha}$

M **3** A quartic equation is given as $x^4 + x + 2 = 0$. It has roots $\alpha, \beta, \gamma, \delta$. State the values of S_1 and S_{-1}, and find the value of S_2.

M **4** The quartic equation $2x^4 + x^3 - x + 7 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $S_3 = \frac{11}{8}$, and using S_n, find the value of S_4.

M **5** You are given that $x^4 - x^3 + x + 2 = 0$, where the roots are $\alpha, \beta, \gamma, \delta$. Find the values of $\Sigma\alpha$, $\Sigma\alpha^2$ and $\Sigma\frac{1}{\alpha}$. Hence, determine the value of $\Sigma\alpha^3$.

M **6** The quartic polynomial $x^4 + ax^2 + bx + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Given that $S_2 = S_{-1}$, find S_3 in terms of a.

M PS **7** The polynomial $3x^4 + 2x^3 + 7x^2 + 4 = 0$ has roots $\alpha, \beta, \gamma, \delta$, where $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$.

a Find the values of S_1 and S_2.

b Find the values of S_3 and S_4.

c Are there any complex roots? Give a reason for your answer.

M **8** For the polynomial $x^4 + ax^3 + bx^2 + c = 0$, with roots α, β, γ and δ it is given that $\alpha + \beta + \gamma + \delta = 2$, $\alpha\beta\gamma\delta = 1$ and $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0$. Find the values of the coefficients a, b and c.

P **9** The roots of the quartic $x^4 - 2x^3 + x^2 - 4 = 0$ are $\alpha, \beta, \gamma, \delta$. Show that $S_4 = 9S_3$.

1.4 Substitutions

Imagine that we are given the quadratic equation $x^2 + 3x + 5 = 0$ with roots α, β and we are asked to find a quadratic that has roots $2\alpha, 2\beta$. There are two approaches that we can adopt.

First, consider the quadratic $(y - 2\alpha)(y - 2\beta) = 0$, then $y^2 - (2\alpha + 2\beta)y + 4\alpha\beta = 0$. If we compare this with the original, which is $\alpha + \beta = -3, \alpha\beta = 5$, then $y^2 + 6y + 20 = 0$ is the new quadratic. This method requires us to know some results, or at least spend time working them out.

A second method is to start with $y = 2x$, since each root of y is twice that of x. Then, substituting $x = \frac{y}{2}$ into the original gives $\left(\frac{y}{2}\right)^2 + 3\left(\frac{y}{2}\right) + 5 = 0$. Alternatively, multiplying by 4, $y^2 + 6y + 20 = 0$. This second approach does not need the values of properties of roots. It just needs the relationship between the roots of each polynomial.

TIP

You learned in AS & A Level Mathematics Pure Mathematics 1 Coursebook how to find inverse functions by switching x and y. This process is helpful for this topic, too.

WORKED EXAMPLE 1.10

Given that $x^2 - 2x + 12 = 0$ has roots α, β, find the quadratic equation with roots $\frac{\alpha}{3}, \frac{\beta}{3}$.

Answer

$y = \frac{x}{3} \Rightarrow x = 3y$ — Rearrange to make x the subject.

$(3y)^2 - 2(3y) + 12 = 0$ — Substitute for x.

$3y^2 - 2y + 4 = 0$ — Cancel factors to simplify.

More complicated substitutions include reciprocal functions. For example, consider the cubic function $x^3 + x^2 - 7 = 0$ with roots α, β, γ. If we are then asked to find a cubic function with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, we would begin with $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$. Then $\left(\frac{1}{y}\right)^3 + \left(\frac{1}{y}\right)^2 - 7 = 0$, which simplifies to the cubic $7y^3 - y - 1 = 0$.

WORKED EXAMPLE 1.11

Given that $x^3 + x^2 - 5 = 0$ has roots α, β, γ, find the cubic equation with roots $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}, \frac{1}{\gamma - 2}$.

Answer

$y = \frac{1}{x - 2} \Rightarrow xy - 2y = 1$

$x = \frac{1+2y}{y}$ — Rearrange to make x the subject.

$\left(\frac{1+2y}{y}\right)^3 + \left(\frac{1+2y}{y}\right)^2 - 5 = 0$ — Substitute for x.

$\frac{1 + 6y + 12y^2 + 8y^3}{y^3} + \frac{1 + 4y + 4y^2}{y^2} - 5 = 0$ — Expand brackets.

$7y^3 + 16y^2 + 7y + 1 = 0$ — Multiply by y^3 and simplify.

EXPLORE 1.1

The polynomial $x^3 + x - 3 = 0$ has roots α, β, γ. If $\frac{a\alpha + 1}{\alpha - b}, \frac{a\beta + 1}{\beta - b}, \frac{a\gamma + 1}{\gamma - b}$ are the roots of another cubic, what are the conditions on a and b to ensure that these cubics are the same?

The type of substitutions that are treated differently are those raised to the power. For example, if we have the cubic equation $2x^3 + 7x^2 - 1 = 0$ with roots α, β, γ and we want to determine the cubic with roots $\alpha^2, \beta^2, \gamma^2$, there are two ways of approaching this.

First, we might want to state that $y = x^2$ and so $x = \sqrt{y}$. Substituting gives $2y^{\frac{3}{2}} + 7y - 1 = 0$. Next, write as $7y - 1 = -2y^{\frac{3}{2}}$ and square both sides, giving $49y^2 - 14y + 1 = 4y^3$. So $4y^3 - 49y^2 + 14y - 1 = 0$ is the cubic that we are looking for.

The second approach is to first rearrange the cubic to $2x^3 = 1 - 7x^2$. Doing this allows us to square both sides and get even powers of x for every term, so $4x^6 = 1 - 14x^2 + 49x^4$. Substituting in $x^2 = y$ gives the same result as before.

Both methods adopt the same approach. Whether we substitute before or after rearranging, we must ensure all terms are appropriate in terms of their power.

WORKED EXAMPLE 1.12

The polynomial $x^4 + x^3 - x + 12 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the polynomial with roots $\alpha^2, \beta^2, \gamma^2, \delta^2$.

Answer

$y = x^2$ — State the substitution.

$x^4 + 12 = x - x^3$ — Rearrange so that both sides when squared give even terms.

$x^8 + 24x^4 + 144 = x^2 - 2x^4 + x^6$ — Square both sides.

$x^8 - x^6 + 26x^4 - x^2 + 144 = 0$ — Simplify.

$y^4 - y^3 + 26y^2 - y + 144 = 0$ — Use $x^2 = y$.

These substitution methods are useful when dealing with problems such as finding the value of S_6 or even S_8.

Consider the quartic $x^4 + x^3 - 5 = 0$. For this polynomial, we would like to determine the value of S_4. The process for finding $S_4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4$ can be time consuming. Now, consider that there is another quartic such that $y = x^2$. If this quartic exists, then for y we would have $S_n = \alpha^{2n} + \beta^{2n} + \gamma^{2n} + \delta^{2n}$. Since we have doubled the power for each root, once we have determined the quartic for y we would need to find only S_2, which is straightforward.

Rewrite the original quartic as $x^4 - 5 = -x^3$, then square both sides to get $x^8 - 10x^4 + 25 = x^6$. Next, replace x^2 with y so that $y^4 - y^3 - 10y^2 + 25 = 0$. Finally, for the new quartic, $S_1 = 1$ and $S_2 = 1^2 - 2 \times (-10) = 21$. Hence, for the original quartic, $S_4 = 21$.

This is an effective method and can save lots of time, particularly for much higher values of n.

TIP

Ensure both sides of the rearranged polynomial will give appropriate powers when the squaring or cubing operation has taken place. For example, $x^3 - 5x + 7 = 0$ with $y = x^2$ would be written as $x^3 - 5x = -7$ to ensure that, when squared, both sides produce only even powers.

If the same equation is used with $y = x^3$, then rearrange to $x^3 + 7 = 5x$ to get all powers of 3 on both sides when cubed.

WORKED EXAMPLE 1.13

The cubic polynomial $x^3 + 5x^2 + 1 = 0$ has roots α, β, γ. Using the substitution $y = x^3$, or otherwise, find the value of S_6.

Answer

$x^3 + 5x^2 + 1 = 0 \quad (1)$ — Label the original equation.

$x^3 + 1 = -5x^2$ — Since we are using x^3, ensure all terms are arranged so when both sides are cubed, they produce powers of 3.

$x^9 + 3x^6 + 3x^3 + 1 = -125x^6$ — Cube both sides.

$\Rightarrow x^9 + 128x^6 + 3x^3 + 1 = 0$ — Simplify.

$\Rightarrow y^3 + 128y^2 + 3y + 1 = 0 \quad (2)$ — Use $x^3 = y$, label the new equation.

$S_1 = -128$	Determine S_1.
$S_2 = (-128)^2 - 2 \times 3 = 16\,378$ for (2)	Substitute for S_2.
Hence, for (1), $S_6 = 16\,378$	State S_6.

DID YOU KNOW?

The term 'polynomials' was not used until the 17th century. Before the 15th century, equations were represented by words, not symbols. A famous Chinese algebraic problem was written: 'Three bundles of good crop, two bundles of mediocre crop, and one bundle of bad crop are sold for 29 dou.' In modern times we would phrase this as $3a + 2b + c = 29$.

EXERCISE 1D

M **1** The quadratic equation $x^2 + 5x + 3 = 0$ has roots α, β. Find the quadratic equation with roots $3\alpha, 3\beta$.

M **2** The quadratic equation $2x^2 - 4x + 7 = 0$ has roots α, β.

a Find the quadratic equation with roots α^2, β^2.

b Find the quadratic equation with roots $2\alpha - 3, 2\beta - 3$.

M **3** Given that $3x^2 - 2x + 9 = 0$ has roots α, β, find the quadratic equation with roots $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$.

PS **4** The quadratic equation $x^2 - 4x + 9 = 0$ has roots α, β. Find the quadratic that has roots $\frac{1}{\alpha}, \frac{1}{\beta}$.

PS **5** Given that $2x^3 - 5x + 1 = 0$ has roots α, β, γ, find the cubic equation with roots $\alpha^2, \beta^2, \gamma^2$. Hence, find the value of S_4.

M P **6** The cubic equation $x^3 + 3x^2 - 1 = 0$ has roots α, β, γ. Show that the cubic equation with roots $\frac{\alpha+2}{\alpha}, \frac{\beta+2}{\beta}, \frac{\gamma+2}{\gamma}$ is $y^3 - 3y^2 - 9y + 3 = 0$. Hence, determine the values of:

a $\frac{(\alpha+2)(\beta+2)(\gamma+2)}{\alpha\beta\gamma}$ **b** $\frac{\alpha}{\alpha+2} + \frac{\beta}{\beta+2} + \frac{\gamma}{\gamma+2}$

M P **7** A quartic equation, $2x^4 - x^3 - 6 = 0$, has roots $\alpha, \beta, \gamma, \delta$. Show that the quartic equation with roots $\alpha^3, \beta^3, \gamma^3, \delta^3$ is $8y^4 - y^3 - 18y^2 - 108y - 216 = 0$. Hence, find the values of S_6 and S_{-3}.

PS **8** The cubic equation $x^3 - x + 4 = 0$ has roots α, β, γ. Find the cubic equation that has roots $\alpha^2, \beta^2, \gamma^2$. Hence, or otherwise, determine the values of S_6, S_8 and S_{10}.

WORKED PAST PAPER QUESTION

The equation $x^3 + x - 1 = 0$ has roots α, β, γ.

Show that the equation with roots $\alpha^3, \beta^3, \gamma^3$ is $y^3 - 3y^2 + 4y - 1 = 0$.

Hence, find the value of $\alpha^6 + \beta^6 + \gamma^6$.

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Answer

Start with $x^3 - 1 = -x$.

Then rewrite this as $(x^3 - 1)^3 = -x^3$. This gives $x^9 - 3x^6 + 4x^3 - 1 = 0$.

Let $y = x^3$ to give $y^3 - 3y^2 + 4y - 1 = 0$.

Note that $S_n = \alpha^n + \beta^n + \gamma^n$.

S_6 for the original equation is S_2 for the new equation, so $S_2 = 3^2 - 2 \times 4 = 1$.

Hence, $\alpha^6 + \beta^6 + \gamma^6 = 1$.

Checklist of learning and understanding

For quadratic equations ($ax^2 + bx + c = 0$):

- $\Sigma\alpha = \alpha + \beta = -\frac{b}{a}$
- $\Sigma\alpha\beta = \alpha\beta = \frac{c}{a}$
- $S_n = \alpha^n + \beta^n$

For cubic equations ($ax^3 + bx^2 + cx + d = 0$):

- $\Sigma\alpha = \alpha + \beta + \gamma = -\frac{b}{a}$
- $\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
- $\Sigma\alpha\beta\gamma = \alpha\beta\gamma = -\frac{d}{a}$
- $S_n = \alpha^n + \beta^n + \gamma^n$

For quartic equations ($ax^4 + bx^3 + cx^2 + dx + e = 0$):

- $\Sigma\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
- $\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
- $\Sigma\alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
- $\Sigma\alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$
- $S_n = \alpha^n + \beta^n + \gamma^n + \delta^n$

For recurrence notation:

- $\Sigma\alpha$ is also known as S_1.
- $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ is also known as S_2.
- $\Sigma\frac{1}{\alpha}$ is known as S_{-1}. It is always equal to the negative of the coefficient of the linear term divided by the coefficient of the constant term.

END-OF-CHAPTER REVIEW EXERCISE 1

1 The roots of the equation $x^3 + 4x - 1 = 0$ are α, β and γ. Use the substitution $y = \frac{1}{1+x}$ to show that the equation $6y^3 - 7y^2 + 3y - 1 = 0$ has roots $\frac{1}{\alpha+1}, \frac{1}{\beta+1}$ and $\frac{1}{\gamma+1}$.

For the cases $n = 1$ and $n = 2$, find the value of $\frac{1}{(\alpha+1)^n} + \frac{1}{(\beta+1)^n} + \frac{1}{(\gamma+1)^n}$.

Deduce the value of $\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}$.

Hence show that $\frac{(\beta+1)(\gamma+1)}{(\alpha+1)^2} + \frac{(\gamma+1)(\alpha+1)}{(\beta+1)^2} + \frac{(\alpha+1)(\beta+1)}{(\gamma+1)^2} = \frac{73}{36}$.

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2 The roots of the quartic equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$ are α, β, γ and δ.

Find the values of

i $\alpha + \beta + \gamma + \delta$,

ii $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$,

iii $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$,

iv $\frac{\alpha}{\beta\gamma\delta} + \frac{\beta}{\alpha\gamma\delta} + \frac{\gamma}{\alpha\beta\delta} + \frac{\delta}{\alpha\beta\gamma}$.

Using the substitution $y = x + 1$, find a quartic equation in y. Solve this quartic equation and hence find the roots of the equation $x^4 + 4x^3 + 2x^2 - 4x + 1 = 0$.

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3 The cubic equation $x^3 - x^2 - 3x - 10 = 0$ has roots α, β, γ.

i Let $u = -\alpha + \beta + \gamma$. Show that $u + 2\alpha = 1$, and hence find a cubic equation having roots $-\alpha + \beta + \gamma, \alpha - \beta + \gamma, \alpha + \beta - \gamma$.

ii State the value of $\alpha\beta\gamma$ and hence find a cubic equation having roots $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$.

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